IES 302: Quiz 3 Solution

In this question, the population contains *N* = 3 members: $\{a_1 = 1, a_2 = 1, a_3 = 3\}$. The size of the random sample is *n* = 2. So, the random sample contains two random variables $\{X_1, X_2\}$.

There are $\binom{N}{n} = \binom{3}{2} = 3$ possible samples. (This is sampling without replacement.)

The members in the samples and the corresponding sample means are listed in the table below:

X_1	X_2	Sample mean $\overline{X} = \frac{1}{2}(X_1 + X_2)$
$a_1 = 1$	$a_2 = 1$	1
$a_1 = 1$	$a_3 = 3$	2
$a_2 = 1$	$a_3 = 3$	2

These three possibilities happen with equal probability. So, the pmf (distribution) of the sample mean is given by

$$\boxed{p_{\overline{X}}(\overline{x}) = \begin{cases} 1/3, & \overline{x} = 1, \\ 2/3, & \overline{x} = 2, \\ 0, & \text{otherwise.} \end{cases}}$$

The expected value of the sample mean is given by

$$\mathbb{E}\left[\overline{X}\right] = \sum_{\overline{x}} \overline{x} p_{\overline{X}}(\overline{x}) = 1 \times \frac{1}{3} + 2 \times \frac{2}{3} = \boxed{\frac{5}{3}}..$$

To find the variance, we first find

$$\mathbb{E}\left[\overline{X}^2\right] = \sum_{\overline{x}} \overline{x}^2 p_{\overline{X}}(\overline{x}) = 1^2 \times \frac{1}{3} + 2^2 \times \frac{2}{3} = \frac{9}{3} = 3$$

Therefore,

$$\mathrm{Var}\left[\overline{X}\right] = \mathbb{E}\left[\overline{X}^2\right] - \left(\mathbb{E}\left[\overline{X}\right]\right)^2 = 3 - \left(\frac{5}{3}\right)^2 = \boxed{\frac{2}{9}}.$$